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If the natural numbers be uniformly distributed this reduces to 0.28. When the mantissas are uniformly distributed it becomes 0.27.

Reasoning as in Problem V, we may prove that when the natural numbers are restricted to 5 places, since 4.5 is the smallest tabular difference consistent with the condition that $c = c'$, the maximum error is $\frac{2.5}{4.5} + 0.50$, or 1.06. When the number is carried to 6 places, the maximum error is $10 . \frac{2.5}{4.5} + 0.5$ or 6.1.

I have taken 100 uniformly distributed 5 place mantissas and obtained their antilogarithms from Gernerth's table, carrying the results to 6 places and also to 5 places. More accurate values were then obtained from a 7-place table. Thus it was found that the average of the errors of the 6-place antilogarithms was 1.07 in units of the sixth place instead of 0.99 as the theory gives; the average of the errors of the 5-place antilogarithms was 0.27 in the fifth place, which accords well with the theory.

SOLUTIONS OF EXERCISES.

24

THE curves of the family

$$\left\{\frac{C}{r}\right\}^n = \cos(p + n\vartheta).$$

where n is a parameter, all pass through a fixed point and cut orthogonally the fixed curve

$$\left\{\frac{r}{c}\right\}^n = \cos n\vartheta,$$

provided $C^n = \frac{1}{2}c^n \cos p$.

[Generalization of 14.] [*Alfred C. Lane*]

SOLUTION.

When $\vartheta = 0$ we have

$$r = C \,/\, \sqrt[n]{}(\cos p) = c \,/\, \sqrt[n]{2},$$

which is constant, since c and n are constants. All curves of the family pass, therefore, through the point whose polar co-ordinates are $c \,/\, \sqrt[n]{2}$, 0.

At the point of concourse of

$$\left(\frac{C}{r}\right)^n = \cos(p + n\vartheta), \quad \left(\frac{r}{c}\right)^n = \cos n\vartheta;$$

$$\left(\frac{C}{c}\right)^n = \cos(p + n\vartheta)\cos n\vartheta;$$

$$\therefore \cos p = \cos p + \cos(p + 2n\vartheta);$$

$$\therefore n\vartheta = 45^\circ - \tfrac{1}{2}p.$$

But at this point we have for the first curve

$$\tan\varphi_1 = \frac{rd\vartheta}{dr} = \cot(p + n\vartheta) = \tan n\vartheta$$

and for the second curve

$$\tan\varphi_2 = \frac{rd\vartheta}{dr} = \cot n\vartheta;$$

the two curves, therefore, intersect orthogonally. [*W. M. Thornton.*]

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THE CUBIC $ax^3 + 3bx^2 + 3cx + d = 0$ has one or three real roots as its discriminant

$$D = (ad - bc)^2 - 4(b^2 - ac)(c^2 - bd)$$

is positive or negative. If D is zero, two roots are in general equal each to

$$\sqrt{\frac{c^2 - bd}{b^2 - ac}}$$

SOLUTION.

Put $v = ax^3 + 3bx^2 + 3cx + d = 0;$

then will

$$v_1 = 3ax^2 + 6bx + 3c,$$
$$v_2 = 2(b^2 - ac)x + bc - ad,$$
$$v_3 = -aD = -a[(ad - bc)^2 - 4(b^2 - ac)(c^2 - bd)].$$

Supposing a, b, c, d to be positive, we shall have when D is positive, provided $b^2 > ac$, the following variations in signs :—

	v	v_1	v_2	v_3	
for $x = -a$	$-$	$+$	$-$	$-$,	two variations,
and for $x = +a$	$+$	$+$	$+$	$-$,	one variation,

under which conditions v has one real root. Suppose the other conditions to remain the same, but D to be negative. In this case, we have

$$\begin{array}{llllll} & v & v_1 & v_2 & v_3 & \\ \text{for } x = -a & - & + & - & + \ , & \text{three variations,} \\ \text{and for } x = +a & + & + & + & + \ ; & \text{no variations;} \end{array}$$

and v has, therefore, three real roots.

If v has equal roots, $v_3 = 0$, and, therefore, $D = 0$;

or, $$(ad - bc)^2 - 4(b^2 - ac)(c^2 - bd) = 0;$$

also, the equation $$v^2 = 2(b^2 - ac)x + bc - ad = 0$$

will give for each of the equal roots

$$x = \frac{ad - bc}{2(b^2 - ac)} = \frac{2\surd[(b^2 - ac)(c^2 - bd)]}{2(b^2 - ac)} = \sqrt{\frac{c^2 - bd}{b^2 - ac}}.$$

If $b^2 < ac$, v will have three real roots when D is positive and one when D is negative. [*Asher B. Evans.*]

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LET O, O' be the centres, r, R the radii of the inscribed and circumscribed circles of a triangle ABC; also let A', B', C' be the feet of the perpendiculars from the vertices on the opposite sides, I their intersection, and ρ the radius of the circle inscribed in $A'B'C'$. Prove

$$OI^2 = 2r^2 - 2R\rho,$$
$$O'I^2 = R^2 - 4R\rho.$$

[*Wm. E. Heal.*]

SOLUTION I.

It is easy to see that AA', BB', CC' are the bisectors of the angles of the triangle $A'B'C'$ and therefore I is the centre of the circle inscribed in $A'B'C'$. Draw IE perpendicular to $A'C'$, OR, $O'R'$ perpendicular to CC', $OS, O'S'$ perpendicular to AB, and OT perpendicular to BC. The triangles $C'IE$, $B'BC$ and $B'CI$, ACC' are similar, and we have

$$C'I : \rho = BC : B'C = a : B'C,$$

$$CI : B'C = AC : C'C = b : \frac{2k}{c} = b : \frac{ab}{2R},$$

where we put $BC = a$, $CA = b$, $AB = c$ and $k =$ area of ABC;

$$\therefore CI \, . \, C'I = 2R\rho.$$

We have $$CI = 2O'S' = 2\left\{R^2 - \frac{C^2}{4}\right\}^{\frac{1}{2}} = \frac{R(a^2+b^2-c^2)}{ab},$$

$$CR = CI + C'I - r = \frac{2k}{c} - r = \frac{(a+b+c)r}{c} - r = \frac{(a+b)r}{c},$$

$$CO^2 = OT^2 + CT^2 = r^2 + (s-c)^2 = \frac{4abr^2}{(b+c-a)(c+a-b)},$$

$$C'O^2 - r^2 = CO^2 - CR^2,$$

$$C'O^2 = CO^2 - (CR^2 - r^2) = CO^2 - \frac{(a+b+c)(a+b-c)r^2}{c^2}.$$

1. From the triangle $C'CO$ we have

$$C'I \,.\, CO^2 + CI \,.\, C'O^2 = C'I \,.\, CI^2 + CI \,.\, C'I^2 + (CI + C'I)\, OI^2;$$

whence $$OI^2 = \frac{C'I \,.\, CO^2 + CI \,.\, C'O^2}{CI + C'I} - CI \,.\, C'I$$

$$= \frac{C'I \,.\, CO^2 + CI \,.\, C'O^2}{CI + C'I} - 2R\rho.$$

But $C'I \,.\, CO^2 + CI \,.\, C'O^2$

$$= C'I \,.\, CO^2 + CI \,.\, CO^2 - \frac{(a+b+c)(a+b-c)r^2}{c^2} \,.\, CI,$$

$$CI + C'I = \frac{2k}{c} = \frac{(a+b+c)r}{c};$$

$$\therefore\ \frac{C'I \,.\, CO^2 + CI \,.\, C'O^2}{CI + C'I} = CO^2 - \frac{(a+b-c)r}{c} \,.\, CI$$

$$= \frac{4abr^2}{(b+c-a)(c+a-b)} - \frac{(a+b-c)(a^2+b^2-c^2)Rr}{abc}$$

$$= 2r^2 + \frac{2(a^2+b^2-c^2)r^2}{(b+c-a)(c+a-b)} - \frac{(a+b-c)(a^2+b^2-c^2)Rr}{abc}$$

But $$\frac{2(a^2+b^2-c^2)r^2}{(b+c-a)(c+a-b)} = \frac{(a+b-c)(a^2+b^2-c^2)Rr}{abc};$$

$$\therefore\ OI^2 = 2r^2 - 2R\rho.$$

2. $$O'I^2 = O'C^2 + CI^2 - 2CI \,.\, CR'$$
$$= R^2 + CI^2 - 2CI(CI + C'I - C'R').$$

But $C'R' = O'S' = \frac{1}{2}CI$;

$$\therefore\ OI^2 = R^2 + CI^2 - 2CI(C'I + \tfrac{1}{2}CI) = R^2 - 2CI \,.\, C'I = R^2 - 4R\rho.$$

Corollary 1. The nine-point circle of ABC is the circumscribed circle $A'B'C'$, its centre N is the middle point of $O'I$, and its radius $R' = \frac{1}{2}R$;

$$\therefore\ IN^2 = \frac{R^2 - 4R\rho}{4} = R'^2 - 2R'\rho,$$

which is the well known expression for the distance between the centres of the inscribed and circumscribed circles.

Corollary 2. We have

$$O'O^2 + OI^2 = 2IN^2 + 2ON^2;$$

whence $$ON = \tfrac{1}{2}R - r = R' - r,$$

which shows that the inscribed circle touches the nine-point circle internally.

[*Wm. E. Heal.*]

SOLUTION II.

We presuppose an acquaintance with the following well-known propositions:—

1. $OO'^2 = R^2 - 2Rr$;
2. The altitudes of a triangle ABC bisect the angles of $A'B'C'$, thus making I the centre of the circle inscribed in $A'B'C'$;
3. The centre N of the circle about $A'B'C'$ (the "nine-points circle") is on the line $O'I$ and $NO' = NI$;
4. The radius of circle $A'B'C'$ is one-half that of circle ABC;
5. The circle inscribed in the triangle ABC touches the circle $A'B'C'$ internally, making the distance between their centres equal to the difference of their radii.

Thus $$NI^2 = \left(\frac{R}{2}\right)^2 - 2\left(\frac{R}{2}\right).\rho, \text{ by } 1, 2, 4;$$

i. e. $$\frac{O'I^2}{4} = \frac{R^2}{4} - R\rho, \text{ by } 3,$$

or $$O'I^2 = R^2 - 4R\rho.$$

Considering the triangle $OO'I$, N being the middle point of $O'I$,

$$OO'^2 + OI^2 = 2\,ON^2 + 2\,NI^2;$$

i. e. $$R^2 - 2Rr + OI^2 = 2\left(\frac{R}{2} - r\right)^2 + \frac{R^2}{2} - 2R\rho,$$

or $$OI^2 = 2r^2 - 2R\rho.$$

[*R. D. Bohannan.*]